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NECESSARY AND SUFFICIENT CONDITIONS FOR THE INTER- CHANGE OF LIMIT AND SUMMATION IN THE CASE OF SEQUENCES OF INFINITE SERIES OF A CERTAIN TYPE.

BY T. H. HILDEBRANDT.

Double sequences and series have been discussed by Pringsheim,* London,† and others. They have treated incidentally the question of interchange of iterated limits. An interchange of limit and infinite summation, although to some extent a question of interchange of limits, is a distinct problem in that summation and limit are different operations. The theorems of this paper derive necessary and sufficient conditions for the interchange of limit in the case of a special type of series, practically that of series of positive terms.

THEOREM I. *Suppose a double sequence of numbers x_{np} ($n = 1, 2, \dots$; $p = 1, 2, \dots$) such that $\Sigma_p |x_{np}|$ ‡ is convergent for every n . Suppose also $L_n x_{np} = x_p$ for every p . Then a necessary and sufficient condition that $\Sigma_p |x_p|$ converge and $L_n \Sigma_p |x_{np}| = \Sigma_p |x_p|$ is that the series $\Sigma_p |x_{np}|$ be uniformly convergent.*

That the condition is *sufficient* even in case the absolute value signs be dropped throughout is well known, a consequence of the theorem on the interchange of double limits.§

On the other hand the condition is *necessary*. Since $\Sigma_p |x_p|$ converges we have for every $e > 0$ a p_e such that if $P \geq p_e$ then:

$$\sum_{p=P}^{\infty} |x_p| \leq e/2.$$

Take a particular value of P , say $p_1 = p_e$. Then from

$$L \sum_n |x_{np}| = \sum_p |x_p| \quad \text{follows} \quad L \sum_{n=p_1}^{\infty} |x_{np}| = \sum_{p=p_1}^{\infty} |x_p|,$$

i. e., for every e there exists an n_e such that if $n \geq n_e$ we have:

$$\left| \sum_{p=p_1}^{\infty} |x_{np}| - \sum_{p=p_1}^{\infty} |x_p| \right| \leq e/2.$$

* Pringsheim, Muench. Ber. (1897), pp. 101-152; Math. Ann., 53 (1900), pp. 289-321.

† London, Math. Ann., 53 (1900), pp. 322-370.

‡ Throughout this paper we shall designate $\sum_{p=1}^{\infty}$ by Σ_p and $L_{n=\infty}$ by L_n .

§ Cf., for instance, Hobson, Theory of Functions, p. 466.

Then if $P \geq p_1 = p_e$ and $n \geq n_e$, we have:

$$\sum_{p=P}^{\infty} |x_{np}| \leq \sum_{p=p_1}^{\infty} |x_{np}| \leq \sum_{p=p_1}^{\infty} |x_p| + e/2 \leq e/2 + e/2 = e.$$

Since there will be a finite number of values of n less than n_e we have: for every $e > 0$ there exists a p_e' such that for $P \geq p_e'$ we have:

$$\sum_{p=P}^{\infty} |x_{np}| \leq e, \quad n < n_e.$$

Hence if p_e'' is the greater of p_e and p_e' we have for every n and for every $e > 0$ there exists a p_e'' such that for $P \geq p_e''$ we have:

$$\sum_{p=P}^{\infty} |x_{np}| \leq e;$$

which is the uniformity of convergence desired.

COROLLARY. If $m > 0$, $\Sigma_p |x_{np}|^m$ is convergent for every n and $L_n x_{np} = x_p$ for every p , then a necessary and sufficient condition that $\Sigma_p |x_p|^m$ be convergent and $L_n \Sigma_p |x_{np}|^m = \Sigma_p |x_p|^m$ is that $\Sigma_p |x_{np}|^m$ be uniformly convergent.

THEOREM II. Suppose $m > 0$, $\Sigma_p |x_{np}|^m$ convergent for every n , and $L_n x_{np} = x_p$ for every p . Then a necessary and sufficient condition that $\Sigma_p |x_p|^m$ be convergent and $L_n \Sigma_p |x_{np}|^m = \Sigma_p |x_p|^m$, is that $L_n \Sigma_p |x_{np}|^m - \Sigma_p |x_p|^m = 0$.

To prove this theorem we make use of the following inequality *

$$(1) \quad \left[\sum_{p=1}^n |a_p + b_p|^m \right]^k \leq \left[\sum_{p=1}^n |a_p|^m \right]^k + \left[\sum_{p=1}^n |b_p|^m \right]^k,$$

where $m > 0$, and $k = 1/m$ if $m > 1$ and $k = 1$ if $m < 1$. This inequality may be extended to n infinite if the series on the right hand side are convergent. We shall refer to this extension as inequality (1').

The condition of the theorem is *necessary*. Since $\Sigma_p |x_p|^m$ is convergent and $L_n \Sigma_p |x_{np}|^m = \Sigma_p |x_p|^m$, we have by the corollary above that $\Sigma_p |x_{np}|^m$ are uniformly convergent, i. e., for every $e > 0$ there exists a p_e such that if $P \geq p_e$, we have:

$$\sum_{p=P}^{\infty} |x_{np}|^m \leq e^{1/k} \quad \text{and} \quad \sum_{p=P}^{\infty} |x_p|^m \leq e^{1/k}.$$

Take $P = p_1$ fixed. For $p = 1, 2, \dots, p_1$, we have, since $L_n x_{np} = x_p$ for every p : for every $e > 0$ there exists an n_e such that if $n \geq n_e$ we have:

$$(2) \quad |x_{np} - x_p|^m \leq (e/p_1)^{1/k}.$$

* Cf. Riess, Math. Ann., vol. 69 (1910), p. 455.

Using the inequality (1') and the fact that $k \geq 1$ for every m , we have:

$$\begin{aligned} \left[\sum_p |x_{np} - x_p|^m \right]^k &\leq \sum_{p=1}^{p_1} \left[|x_{np} - x_p|^m \right]^k + \left[\sum_{p=p_1}^{\infty} |x_{np} - x_p|^m \right]^k \\ &\leq e + \left[\sum_{p=p_1}^{\infty} |x_{np}|^m \right]^k + \left[\sum_{p=p_1}^{\infty} |x_p|^m \right]^k \leq 3e, \end{aligned}$$

i. e., for every $e > 0$ there exists an n_e , viz., the n_e needed for inequality (2), such that if $n \geq n_e$, we have:

$$\left[\sum_{p=1}^{\infty} |x_{np} - x_p|^m \right]^k \leq e,$$

in other words:

$$L \sum_p |x_{np} - x_p|^m = 0.$$

The condition is *sufficient*.* Since $L_n \sum_p |x_{np} - x_p|^m = 0$, for every $e > 0$ there will exist an n_e such that if $n \geq n_e$ we have:

$$(3) \quad \sum_p |x_{np} - x_p|^m \leq e.$$

Let $x_{np} - x_p = e_{np}$. Then $x_p = x_{np} + e_{np}$ and $x_{np} = x_p - e_{np}$. Applying inequality (1'), we obtain:

$$\left[\sum_p |x_p|^m \right]^k = \left[\sum_p |x_{np} + e_{np}|^m \right]^k \leq \left[\sum_p |x_{np}|^m \right]^k + \left[\sum_p |e_{np}|^m \right]^k$$

and

$$\left[\sum_p |x_{np}|^m \right]^k = \left[\sum_p |x_p - e_{np}|^m \right]^k \leq \left[\sum_p |x_p|^m \right]^k + \left[\sum_p |e_{np}|^m \right]^k.$$

From the first of these inequalities we conclude that $\sum_p |x_p|^m$ is convergent.†

From the two inequalities taken together and the condition $n \geq n_e$, which gives us inequality (3), we have:

$$\left[\sum_p |x_{np}|^m \right]^k - e^k \leq \left[\sum_p |x_p|^m \right]^k \leq \left[\sum_p |x_{np}|^m \right]^k + e^k$$

and so

$$L \sum_p |x_{np}|^m = \sum_p |x_p|^m.$$

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* For $m \leq 2$ this is a consequence of a theorem due to Hilbert, Goett. Nach. Math. Phys. Klasse (1906), p. 177.

† Also a consequence of Moore, General Analysis (Yale Coll. Lect.) §16, p. 38.